A method to find the volume of a sphere in the Lee metric, and its applications

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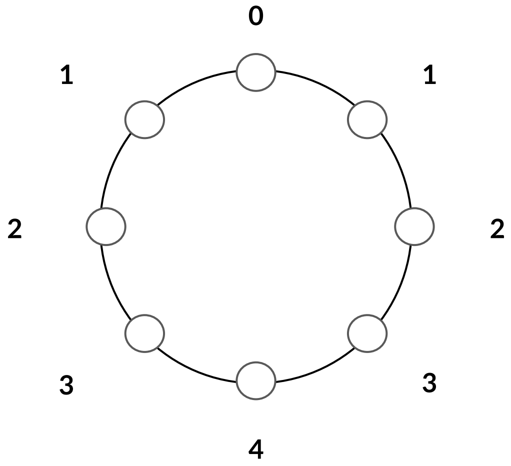

Question

How do we find bounds on the size of codes for discrete metrics other than the Hamming metric? What about the Lee metric?

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- Berlekamp[2] also gave the EB bound for the Lee metric

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But the **parameter regime is too small**.

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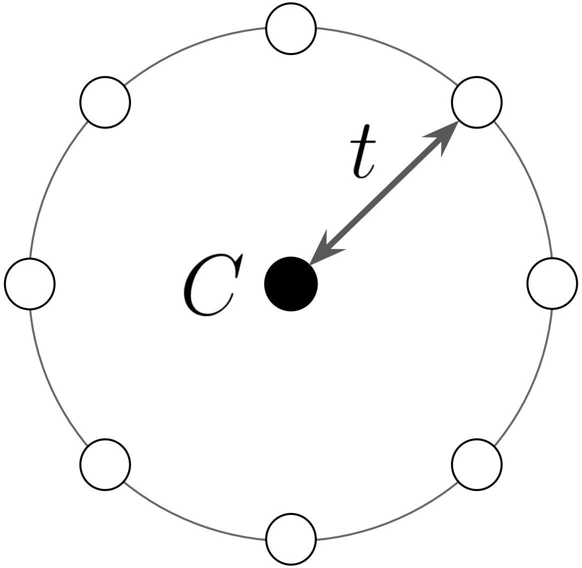
- It reduces to the known results for the Hamming metric
- It allows us to find bounds on the rate for the Lee metric

Our Methods

$$A_t^{(n)} := |\{x \in [q]^n : \mathrm{dist}(C, x) = t\}|$$



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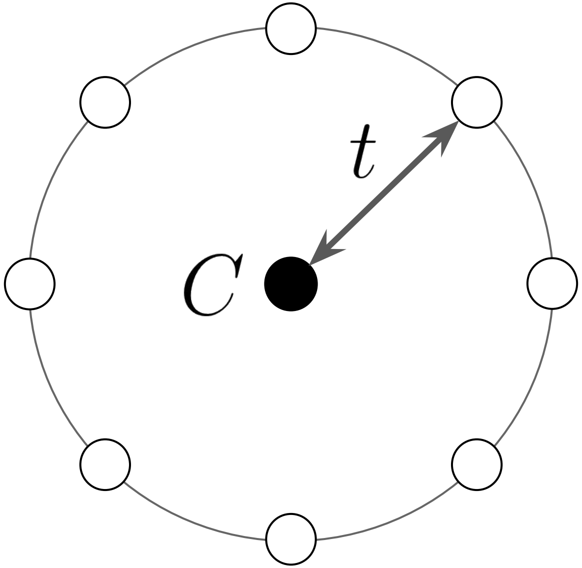

$$V_t^{(n)} = \sum_{j=0}^{t} A_j^{(n)}$$

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The Generating Function

- The generating function for the $A_j^{(n)}$

$$A^{(n)}(z) = \sum_j A_j^{(n)} z^j$$

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$$A^{(1)}(z) = 1+2z+2z^2+\ldots+2z^{\frac{q-2}{2}}+z^{\frac{q}{2}}$$

Lee metric for even q

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New Question

How do we find

$$\sum_j A_j^{(n)} \text{ ?}$$

Involving a probability distribution

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Which defines a discrete random variable X that takes value *0* w.p. *1/q*, *1* w.p. *2/q* and so on.

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The $B_j^{(n)}$ give the probability that the sum of n *i.i.d.* samples drawn according to X add up to j

Sanov's Theorem

Theorem 1 (Sanov's Theorem)**.** *Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\sim Q(x)$. Let $E \subseteq \mathcal{P}$ be a set of probability distributions and $\mathcal{P}$ be the set of all types from the n realisations $X_1, X_2, \ldots, X_n$. Then,*

$$Q^n(E) = Q^n(E \cap \mathcal{P}_n) \leq (n+1)^{|\mathcal{X}|} 2^{-nD(P^*||Q)}$$

where $|\mathcal{X}|$ is the support of each X_i, $D(\cdot||\cdot)$ is the K-L divergence, $Q^n(E)$ is the probability that the empirical distribution obtained from an n-long sample $X_1, \ldots, X_n$ each $\sim Q(x)$ belongs to the set E, and

$$P^* = \arg\min_{P \in E} D(P||Q)$$

is the distribution in E that is closest to Q in relative entropy. If we also have that the set E is the closure of its interior, then we also have the result

$$\frac{1}{n} \log Q^n(E) \to -D(P^*||Q)$$

Retaining terms upto first order in the exponent, we have

$$2^{-nD(P^*||Q)-o(n)} \leq Q^n(E) \leq 2^{-nD(P^*||Q)+o(n)}$$

Sanov, I. N. (1957) "On the probability of large deviations of random variables"
Cover, Thomas M.; Thomas, Joy A. (2006). Elements of Information Theory (2 ed.)

Using Sanov's theorem - finding E

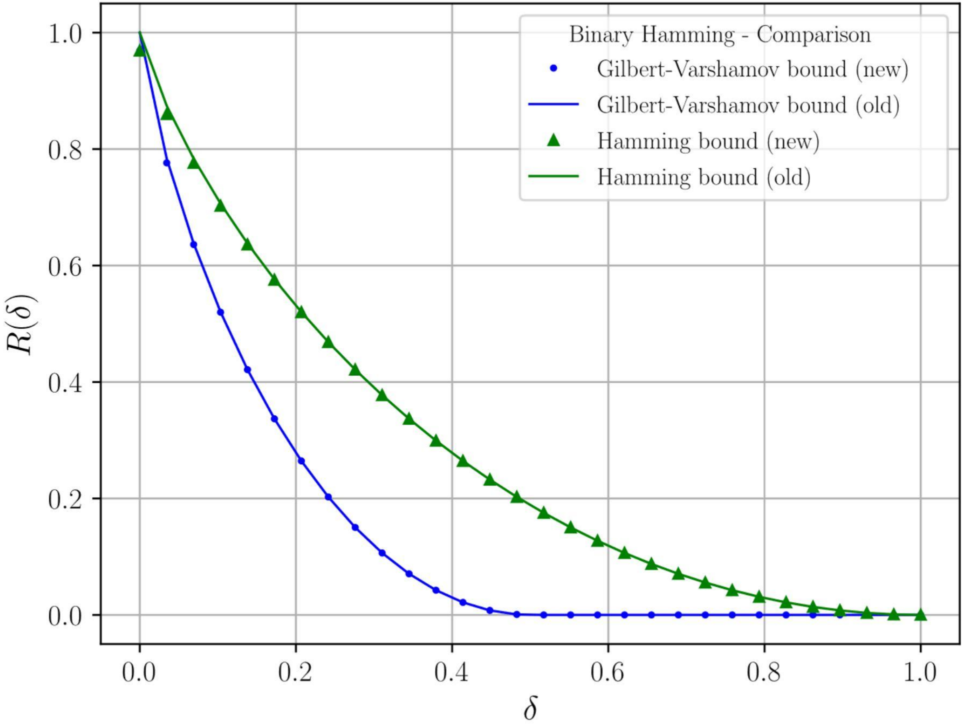
The natural choice while calculating the k^{th} coefficient is the set of all distributions with mean less than or equal to k.

Sanity Check - Hamming metric

The Hamming metric does not require convex optimisation. The random variable in the Hamming case is **Bernoulli**, and the KL divergence minimising distribution is not hard to find. The result is familiar.

$$q^{nH_q(p)-o(n)} \leq V_{pn}^{(n)} \leq q^{nH_q(p)+o(n)}$$

Sanity check for the Hamming metric



Convex Optimisation

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Strong duality holds for the problem, so any solution to the dual implies an upper bound for the primal.

Convex optimisation - functional form

The dual program is given by

$$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & -p\lambda - \log\left(\sum_j \mathbb{P}_X(j)e^{-j\lambda}\right) \\ \text{subject to} \quad & \lambda \geq 0 \end{aligned}$$

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Since **any** $\lambda(p)$ will give an upper bound by strong duality, we can choose the function to be $\lambda(p) = c(q)(\overline{D}^{\frac{1}{q}} - p^{\frac{1}{q}})$

Functional form - intuition

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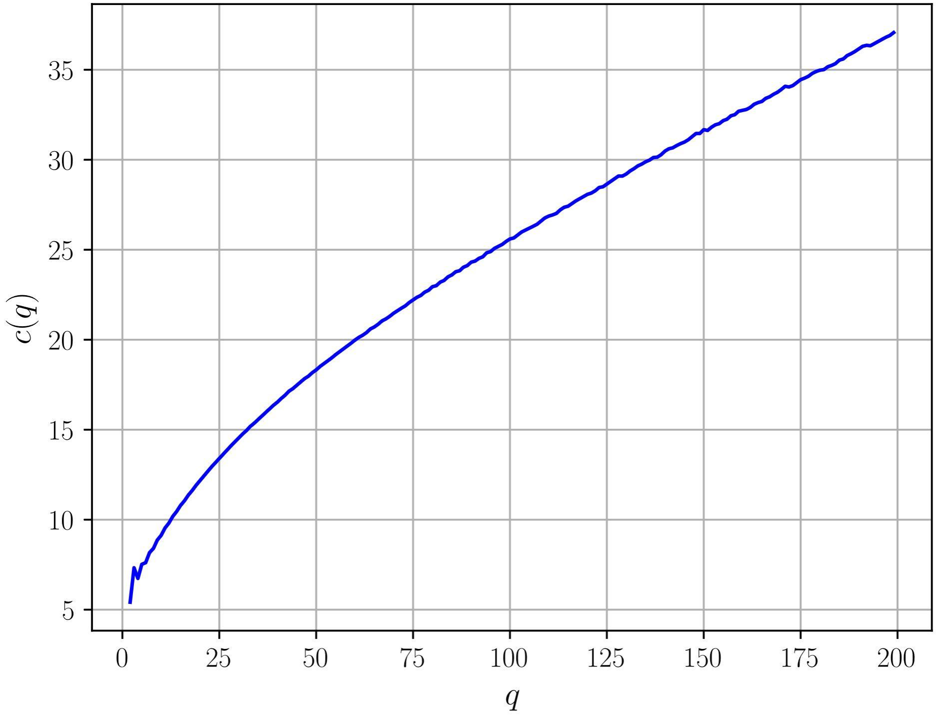
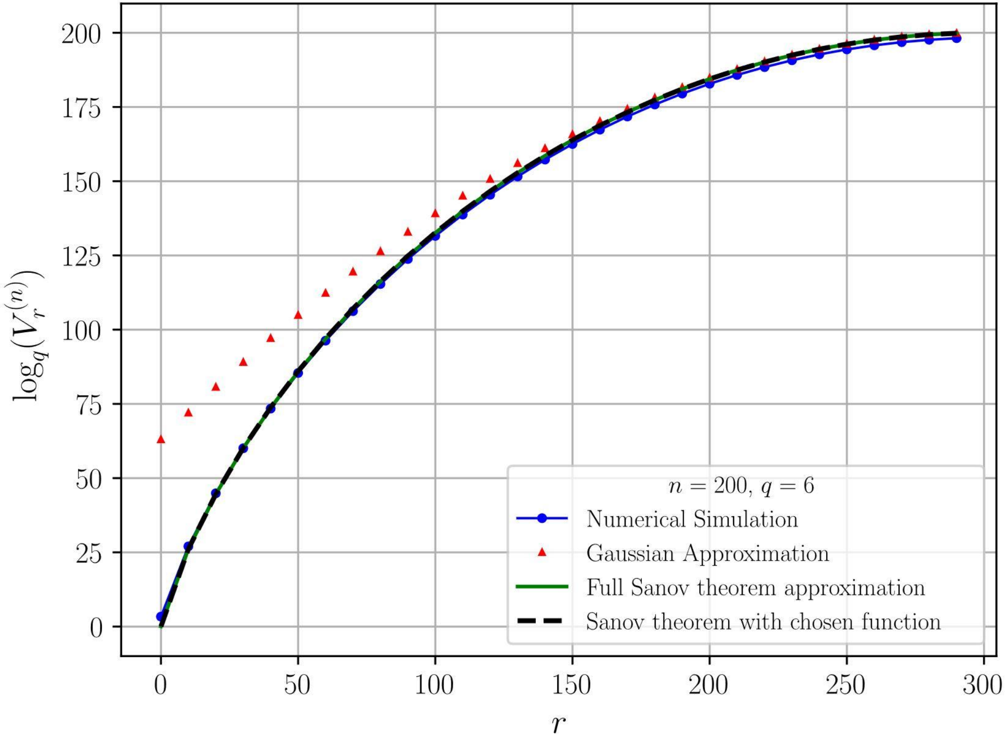
Open - more analytical justification of why this is the form.

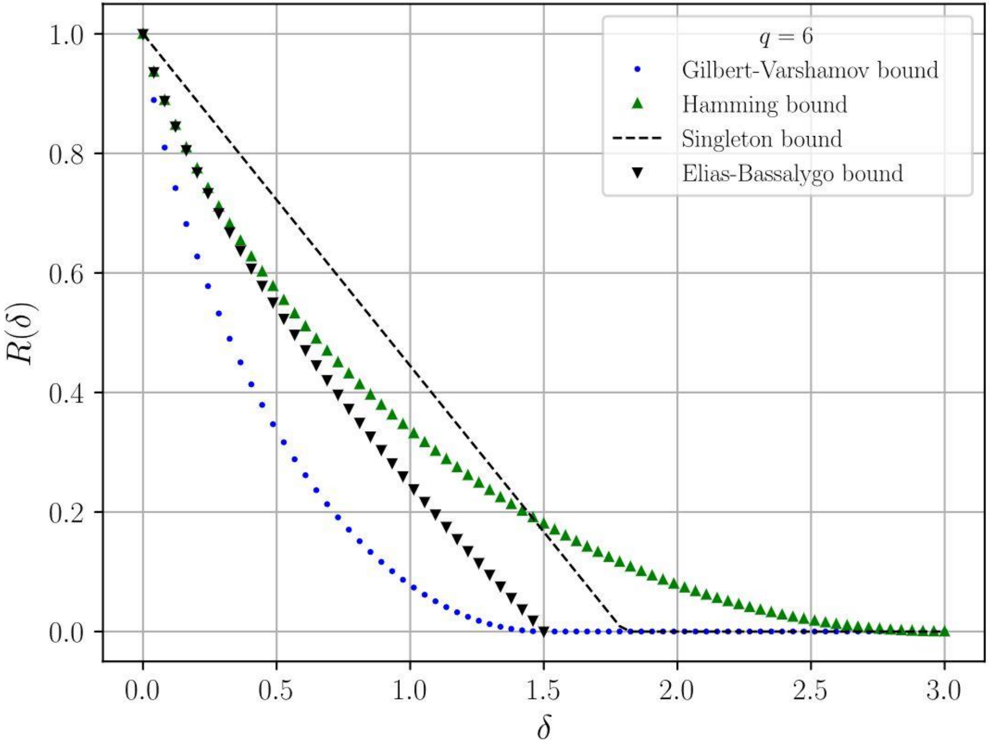
What is the value of $c(q)$?



Immediate result - asymptotic sizes of Lee balls



Also - bounds on codes in Lee metric



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- Method generalises to all discrete metrics with the following property - the set of distances of all symbols to one fixed symbol remains the same when the fixed symbol is replaced by some other symbol
- Should generalise to other discrete metrics too, but the expressions would be more complicated.

Thank you!