$C_0$ = least upper bound on rates at which it is possible to transmit information with zero probability of error.

matrix $\| P_i(j) \|$ $\quad i \longrightarrow j$ $\quad \sum_j P_j(i) = 1$

sequence of input letters = word

rate = $R = \frac{1}{n} \log M$

a code maps $\{1, \ldots, M\} \rightarrow \{\text{words of length } n\}$

two input letters are adjacent if $\exists$ an output letter that can be caused by both ie $\exists\ t$ s.t $P_i(t) \neq 0$ $P_j(t) \neq 0$.

If everything is adjacent then $0 \neq P_e \geq \frac{M-1}{M} P_{min}^n$

$P_{min} \leftarrow$ smallest non-zero $P_i(j)$

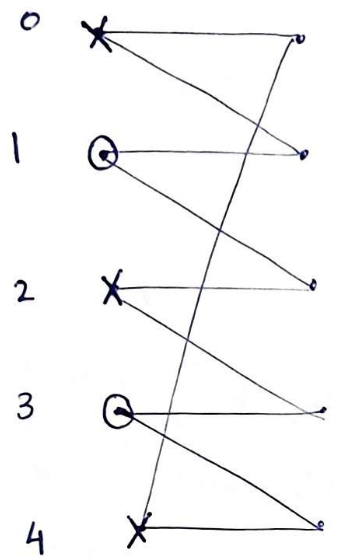
If not everything is adjacent, zero-error is possible.

$M_0(n) \leftarrow$ largest number of possible words with 'block-length' $n$ then clearly

$$\text{lub } \frac{1}{n} \log M_0(n) = C_0$$

$C_0 \neq \log M_0(1)$ in general.

for example consider

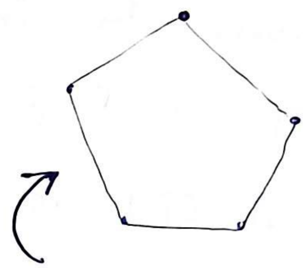



in length 2, 00, 12, 24, 31, 43 $\Big\}\ 5 = M_0(2)$

$\Rightarrow C_0 \geq \frac{1}{2} \log 5.$

adjacency/ confusability diagram.

$\boxed{\text{Pentagon}}$



adjacency graph

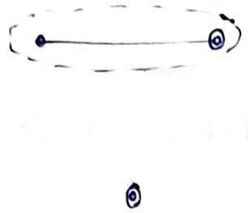
<u>Adjacency reducing mapping</u>.

$i \longrightarrow \alpha(i)$

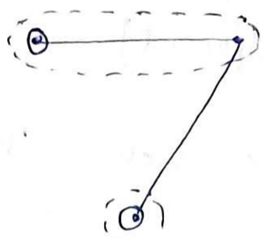
$i$ and $j$ are not adjacent $\Rightarrow \alpha(i)$ and $\alpha(j)$ are also not

<u>Theorem</u> If all i/p letters $i$ can be mapped into a subset of the letters, no two of which are adjacent, then

$$C_0 = \log \left[ \#\text{ of letters in subset} \right]$$



2



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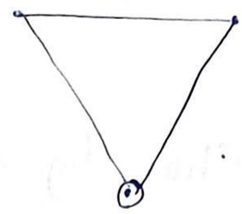
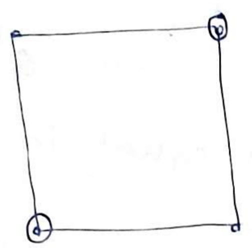


1



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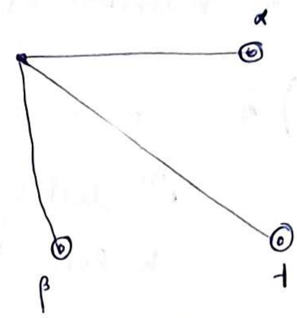
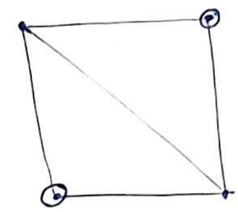



3



2

$$\tfrac{1}{2}\log 5 \leq \text{pentagon} \leq \log(5/2)$$

consider best DMC etc.

For the rest of the talk, we will concentrate on finding the 'Shannon capacity of the pentagon' or $C_0$ for a channel with that adjacency graph.

Open problem for several years

Fast forward to Lovasz – solves the problem, proposes technique that solves several other problems.

Shannon's last result can be stated as -

$\Theta(G) = \alpha(G)$ if $G$ can be covered with $\alpha(G)$ cliques. Cliques marked in diagram above.

[$\alpha(G)$ → introduced next page]

$\alpha(G) \rightsquigarrow$ number of independent points. (set where no two are adjacent).

> Show by an example how Shannon's statement is the same as this

Now, $\alpha(G) \leftarrow$ number of independent pts in $G$.

$\alpha(G^k) \leftarrow$ " " " " " $G^k$

$i^{th}$ letter confoundable or equal in a k-long block.

We showed (by the example of a pentagon) that

$$\alpha(G^k) \neq \alpha(G)^k \text{ in general.}$$

In fact, clearly,

$$\alpha(G^k) \leq \alpha(G^k)$$

[new non-adjacent words may be introduced by the powering.

$\Theta(G) \leftarrow$ Shannon capacity

$$= \sup_k \sqrt[k]{\alpha(G^k)}$$

$$= \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)}$$

$$\geq \alpha(G)$$

> We want to somehow calculate $\Theta(G)$ for the pentagon and $\log \Theta(G) = C_0$

Define product of two graphs $V(G \cdot H) = V(G) \times V(H)$

$(x, y) \sim (x', y')$ iff $x \sim x'$ and $y \sim y'$

$G^k \leftarrow$ k-copies.

Define <u>tensor product</u> of two vectors.

$$v = (v_1 \cdots v_n)^T$$
$$w = (w_1 \cdots w_m)^T$$

$$v \circ w = (v_1 w_1, \ldots, v_1 w_m, \ldots, v_n w_m)^T$$

$$(*) - (x \circ y)^T (v \circ w) = (x^T v)(y^T w)$$ easy to see!

Given a graph $G$, an <u>orthonormal representation</u> is

$\underbrace{(v_1, \ldots v_n)}_{n\text{-vectors}}$ $i$ and $j$ are non-adjacent $\Rightarrow v_i \perp^r v_j$ $(**)$

(for any graph with $n$ vertices, $n$ vectors are sufficient, because choose basis vectors of an $n$-dim space)

If $(u_1, \ldots, u_n)$ is an orthonormal rep for $G$

and $(v_1 \ldots v_m)$ " " " " $H$

then $\{u_i \cdot v_j\}$ is an orthonormal rep for $G \cdot H$

by using $(*)$ and noting that the requirement $(**)$ is one way.

Value of orthonormal representation is defined as

$$\min_{c} \max_{1 \leq i \leq n} \frac{1}{(c^T u_i)^2} \qquad |c| = 1.$$

the ~~xxxxxx~~ $\underline{c}$ that achieves this value is called the handle of the representation.

$\theta(G) \Rightarrow$ min value over all representations

small theta $\Rightarrow$ $\boxed{\text{Lovasz theta function}}$

representation is called optimal if it achieves this minimum.

<u>Lemma</u> $\boxed{\theta(G \cdot H) \leq \theta(G) \cdot \theta(H)}$

Let G has optimal rep $(u_1, \ldots, u_n)$ with handle c.
H " " " $(v_1, \ldots v_m)$ " " d.

$c \cdot d \Rightarrow$ unit vector ~~xxxxxx~~.

by *

$$\theta(G \cdot H) \leq \max_{i,j} \frac{1}{((c \cdot d)^T (u_i \cdot v_j))^2}$$

$\boxed{\because c \cdot d \text{ may be non-optimal}}$

$\boxed{\text{get rid of the min, so the inequality follows}}$

$$= \max_{i,j} \frac{1}{(c^T u_i)^2 (d^T v_j)^2} \quad \text{using } *$$

$$= \max_{i} \frac{1}{(c^T u_i)^2} \max_{j} \frac{1}{(d^T v_j)^2}$$

$$= \theta(G)\, \theta(H)$$

**Theorem** $\alpha(G) \leq \theta(G)$.

**Proof** Let $(u_1, \ldots, u_n)$ a optimal representation with handle $c$.

$\{1, \ldots, k\}$ maximal independent set.

$\Rightarrow \{u_1, \ldots, u_k\}$ are pairwise orthogonal by definition

$$1 = |c|^2 \geq \sum_{i=1}^{k} (c^T u_i)^2$$

$u_i$'s form a subset of an orthogonal basis and $c^T u_i$'s are projections

$$\geq \sum_{i=1}^{k} (c^T u^*)^2$$

$u^*$ minimizes $c^T u_i$

$$= \frac{k}{1/(c^T u^*)^2}$$

$u^*$ maximizes this

$$= \frac{\alpha(G)}{\theta(G)}.$$

Theorem $\quad \Theta(G) \leq \vartheta(G)$.

$$\alpha(G^k) \leq \vartheta(G^k) \leq (\vartheta(G))^k$$

$$\Rightarrow \sqrt[k]{\alpha(G^k)} \leq \vartheta(G)$$

$$\Rightarrow \underbrace{\Theta(G)}_{\text{Shannon-capacity}} \leq \underbrace{\vartheta(G)}_{\text{lovasz Theta}}.$$

We already know $\quad \Theta(G) \geq$ ~~[illegible]~~ $\sqrt{5}$ because

$C_0 \geq 1/2 \log 5$.

we can give an orthonormal rep. for the pentagon

(practical demo using umbrella here!)

which has value $\sqrt{5}$.

$$\therefore \sqrt{5} \leq \Theta(G) \leq \sqrt{5}.$$

$$\therefore \Theta(G) = \sqrt{5} \quad \text{(proved!)}$$