Unit Quantum Protocols and their Resource Theory

End Term Presentation - CS682

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Outline

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  - More specifically, we will consider
    - entanglement generation
    - classical coding
    - super-dense coding
    - quantum teleportation (seen in class)

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- unit resources if it comes in a form such as classical bits, qubits, entangled bits.
- Example: any map of the form $|i\rangle_A \to |i\rangle_B$ is a noiseless qubit channel, and is a non-local unit resource

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- A statement like this means that $\exists$ at least one protocol that uses the resource on the left to give the resource on the right
- We will soon see an easy way to generate entanglement using a quantum channel, so we know that $[q \to q] \geq [qq]$. Relativistic arguments show that the converse is not true.
- Therefore, the quantum channel is the strongest channel.

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- **Note:** We have only called it an ebit after the transfer takes place. Before that, we have just a Bell state, which is not non-local.

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Optimality

Can we send more than one classical bit with one use of the quantum channel?

Teleportation
Quick Recap

The protocol requires the use of entanglement. Alice and Bob can meet before and keep one part (qubit) of the Bell state with each of them. Suppose Alice wants to transfer state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob.

Suppose the state Alice wants to transfer, $|\psi\rangle$, is the first qubit and her part of Bell state is the second qubit. Alice applies CNOT gate to these two qubits. CNOT gate is a 2-qubit gate, which applies NOT gate to the second qubit if and only if the first qubit is in state $|1\rangle$.

Exercise 22. Write the matrix representation of CNOT. Show that CNOT is unitary.

Then she applies Hadamard gate to her first qubit.

Exercise 23. What is the state of the three qubits now?

It can be shown that the resulting state is,

$$\frac{1}{2}\left(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\right).$$

Now Alice measures her two qubits and sends them to Bob.

Exercise 24. Convince yourself that Bob can recover $|\psi\rangle$ using Pauli operators.

This completes the quantum teleportation. Alice is able to transfer one quantum bit using two classical bits of communication.

Teleportation

The resource inequality

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What is the resource inequality?

The resource inequality can be seen to be

$$[qq] + 2\,[c \to c] \geq [q \to q]$$

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Resource Inequality

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We can then keep repeating this and achieve unbounded amount of quantum communication, which is impossible. $\square$

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We can repeat this to obtain $[q \to q] + \infty\,[qq] \geq C^k\,[q \to q] + \infty\,[qq]$, which is impossible. $\square$

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Example

Superdense coding corresponds to the point $x_{SD} := (2, -1, -1)$. Entanglement generation corresponds to the point $x_{EG} := (0, -1, 1)$. Similarly for teleportation. $x_T := (-2, 1, -1)$

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This can be shown by noting that

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

gives all achievable triples along with the fact that the α, β, γ cannot be negative.

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We skip the proof of this theorem. The direct coding theorem which shows that $\widetilde{C_U} \subseteq C_U$ follows immediately from the definitions. The converse coding theorem is non-trivial to prove. Optimality of the protocols follows from the theorem proved.

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- We have shown that the teleportation, super-dense coding and entanglement generation protocols are optimal.
- We have seen a Shannon-like proof of the 'universality' of the three protocols.