Noisy Quantum Theory and Quantum Channels

Mid Term Presentation - CS682

Sagnik Bhattacharya
Instructor: Prof Rajat Mittal

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Outline

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What is the density matrix?

- Defined as the following:

$$\rho := \sum_{i \in \mathcal{X}} p_X(i) \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

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- Treat it as an ensemble of quantum states. In general a single density matrix could correspond to more than one ensemble.

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- **Evolution** Given a unitary U the evolved density matrix is $\rho' = U\rho U^\dagger$
- **Measurement** Given a projection operator Π_i the post-measurement state is given by $\frac{\Pi_i \rho \Pi_i}{Tr\{\rho \Pi_i\}}$

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Ensemble of Ensembles

One more step of uncertainty

- An ensemble of density matrices. It has a density matrix representation given by

$$\rho = \sum_{i \in \mathcal{X}} p_X(i) \rho_i$$

Composite States

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- For multiple Hilbert spaces, simple tensor product analogues to the results discussed also hold. For example, the joint density matrix for two independent systems with density matrices ρ and σ respectively is given by $\rho \otimes \sigma$
- Definitions of product states, separable states and entangled states are similar to those used for the composite noiseless case.
- A partial trace is also defined, to consider a subset of multiple Hilbert spaces.

The Classical-Quantum Ensemble

We generalise one final time to the Classical-Quantum Ensemble, given by

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The $|x\rangle$s form an orthonormal basis It is called Classical-Quantum because it is like a classical state tensored with a quantum state. Why is it useful?

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- $\mathcal{N}$ is a convex linear map that takes density operators in $\mathcal{H}_A$ to density operators in $\mathcal{H}_B$
- We can uniquely extend such a map to all linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$.

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  - **The channel must be linear**.
  - **The channel must be completely positive.** $I_R \otimes \mathcal{M}$ preserves positive semi-definite operators on $\mathcal{L}(\mathcal{H})$ for a reference system R of arbitrary size.
  - **The channel must preserve trace.**

Choi-Kraus Theorem

A map in $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ is a quantum channel iff it has a Choi Kraus Decomposition.

$$\mathcal{N}(X_A) = \sum_{l=0}^{d-1} V_l X_A V_l^\dagger$$

where

$$X_A \in \mathcal{L}(\mathcal{H}_A)$$

$$V_l \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$$

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The Choi operator

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- We saw the most general form of a quantum channel that allows us to mathematically formalise the notion of a quantum evolution.
- Outlook
  - The purified theory, that allows us to consider noise as an entanglement effect.
  - The resource framework, that allows us to prove the optimality of certain quantum protocols.